$$\frac{dy}{dx} = \frac{y^2 - x^2}{xy}$$

$$xy\,dy = (y^2 - x^2)\,dx$$

$$\underbrace{(x^2 - y^2)}_{M}\,dx + \underbrace{xy}_{N}\,dy = 0$$

LET $\cancel{x = vy}$ $y = vx$

$$dx = y\,dv + v\,dy$$

$$\cancel{(v^2y^2 - y^2)\,dx + vy^2\,dy = 0}$$

$$\cancel{(v^2 - 1)\,dx + v\,dy = 0}$$

$$\ln|v| + \frac{1}{2}v^{-2} = -\ln|y| + C$$

$$v = \frac{x}{y}$$

$$\cancel{v = ?}$$

$$\# \ln\left|\frac{x}{y}\right| + \frac{1}{2}\frac{y^2}{x^2} = -\ln|y| + C$$

$$\# (\ln|x| - \ln|y|) + \frac{y^2}{2x^2} = -\ln|y| + C$$

$$\# \ln|x| \cancel{+2\ln|y|} + \frac{y^2}{2x^2} = C$$

CONT'D 2 PAGES AFTER

I.F. TO MAKE DE EXACT?

I.F. OF FORM $x^a y^b$?

$$M(tx, ty) = (tx)^2 - (ty)^2 = t^2(x^2 - y^2) = t^2 M(x, y)$$

$$N(tx, ty) = (tx)(ty) = t^2(xy) = t^2 N(x, y)$$

$$(v^2y^2 - y^2)(y\,dv + v\,dy) + vy^2\,dy = 0$$

$$(v^2 - 1)(y\,dv + v\,dy) + v\cancel{y^2}\,dy = 0$$

$$(v^2 - 1)y\,dv + (v^3 - v + v)\,dy = 0$$

$$(v^2 - 1)y\,dv + v^3\,dy = 0$$

$$(v^2 - 1)y\,dv = -v^3\,dy$$

$$\left(\frac{v^2 - 1}{v^3}\right)dv = -\frac{1}{y}\,dy$$

$$\int\left(\frac{1}{v} - \frac{1}{v^3}\right)dv = -\int\frac{1}{y}\,dy$$

$$(x^2-y^2)\,dx + xy\,dy = 0$$

LET $y = vx$

$dy = v\,dx + x\,dv$

$$(x^2 - v^2x^2)\,dx + vx^2\cancel{y}(v\,dx + x\,dv) = 0$$

$$(1-v^2)\,dx + v(v\,dx + x\,dv) = 0$$

$$(\quad 1 \quad)\,dx + (\quad vx \quad)\,dv = 0$$

$$dx = -vx\,dv$$

$$\int -\frac{1}{x}\,dx = \int v\,dv$$

MANDATORY CHECKPOINT: SEPARABLE

$$-\ln|x| + C = \frac{1}{2}v^2$$

$$-2\ln|x| + C = v^2$$

$$v = \pm\sqrt{C - 2\ln|x|}$$

$$y = vx = \pm x\sqrt{C - 2\ln|x|}$$

$$\cancel{4}\ln|x| + \cancel{2\ln|y|} + \frac{y^2}{2x^2} = C$$

$$\frac{y^2}{2x^2} = C - \ln|x|$$

$$y^2 = 2x^2(C - \ln|x|)$$

$$y^2 = x^2(C - 2\ln|x|)$$

$$y = \pm x\sqrt{C - 2\ln|x|}$$

BERNOULLI EQUATIONS

$$\underbrace{\underbrace{\frac{dy}{dx} + p(x)y}_{\text{LHS OF LINEAR ODE}} = g(x)y^n}_{\text{LINEAR ODE}}$$

CASE 1: $n=0$

$$\frac{dy}{dx} + p(x)y = g(x) \quad \text{LINEAR}$$

CASE 2: $n=1$

$$\frac{dy}{dx} + p(x)y = g(x)y$$

$$\frac{dy}{dx} + (p(x) - g(x))y = 0 \quad \text{LINEAR}$$

$$\frac{dy}{dx} = (g(x) - p(x))y$$

$$\frac{1}{y}dy = (g(x) - p(x))dx \quad \text{SEPARABLE}$$

CASE 3: $n \neq 0, 1$

LET $\boxed{v = y^{1-n}}$

$$\frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = (1-n)y^{-n}\frac{dy}{dx} \longrightarrow \frac{dy}{dx} = \frac{1}{1-n}y^{n}\frac{dv}{dx}$$

$$\frac{dy}{dx} + p(x)y = g(x)y^{n}$$

$$(1-n)y^{-n}\left(\frac{1}{1-n}y^{n}\frac{dv}{dx} + p(x)y\right) = \left(g(x)y^{n}\right)(1-n)y^{-n}$$

$$\frac{dv}{dx} + (1-n)p(x)y^{1-n} = (1-n)g(x)$$

$$\frac{dv}{dx} + (1-n)p(x)v = (1-n)g(x) \leftarrow \text{LINEAR}$$

DON'T USE THIS FORMULA AS A SHORTCUT

SOLVE $y' + \frac{y}{x} = x^2 y^2$

$y' + \frac{1}{x} y = x^2 y^2$

$n=2$

$v = y^{1-2} = y^{-1}$ OR $y = v^{-1}$

$-y^{-2}\left(-y^2 \frac{dv}{dx} + \frac{1}{x} y\right) = (x^2 y^2)(-y^{-2})$

$\frac{dv}{dx} = \frac{dy^{-1}}{dy}\frac{dy}{dx}$

$\frac{dv}{dx} = -y^{-2}\frac{dy}{dx}$

$-y^{2}\frac{dv}{dx} = \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{dv^{-1}}{dv}\frac{dv}{dx}$

$\frac{dy}{dx} = -v^{-2}\frac{dv}{dx}$

$= -(y^{-1})^{-2}\frac{dv}{dx}$

$= -y^{2}\frac{dv}{dx}$

$\frac{dv}{dx} - \frac{1}{x}(y^{-1}) = -x^2$

$\frac{dv}{dx} - \frac{1}{x} v = -x^2$

MANDATORY CHECKPOINT: LINEAR

$\mu = e^{\int -\frac{1}{x} dx} = e^{-\ln|x|} = x^{-1}$

$x^{-1}\frac{dv}{dx} - x^{-2} v = -x$

MANDATORY CHECKPOINT:

$(x^{-1})' = -x^{-2}$ ✓

$\frac{d}{dx}(x^{-1} v) = -x$

$x^{-1} v = \int -x\,dx + C$

$x^{-1} v = -\frac{1}{2}x^2 + C$

$v = -\frac{1}{2}x^3 + Cx$

$y^{-1} = -\frac{1}{2}x^3 + Cx$

$y = \frac{1}{Cx - \frac{1}{2}x^3} \cdot \frac{2}{2}$

$$\boxed{y = \frac{2}{Cx - x^3}}$$

2.5
#20

$$3(1+t^2)\frac{dy}{dt} = 2ty(y^3-1)$$

$$3(1+t^2)\frac{dy}{dt} = 2ty^4 - 2ty$$

$$3(1+t^2)\frac{dy}{dt} + 2ty = 2ty^4$$

$n=4$

$V = y^{1-4} = y^{-3}$ OR $y = v^{-\frac{1}{3}}$

$$\frac{dy}{dt} + \frac{2t}{3(1+t^2)}y = \frac{2t}{3(1+t^2)}y^4$$

$$\frac{dv}{dt} = \frac{dy^{-3}}{dy}\frac{dy}{dt}$$

$$-3y^{-4}\left(-\frac{1}{3}y^4\frac{dv}{dt} + \frac{2t}{3(1+t^2)}y\right) = \left(\frac{2t}{3(1+t^2)}y^4\right)(-3y^{-4})$$

$$\frac{dv}{dt} = \underline{-3y^{-4}}\frac{dy}{dt}$$

$$\frac{dy}{dt} = -\frac{1}{3}y^4\frac{dv}{dt}$$

$$\frac{dv}{dt} - \frac{2t}{1+t^2}y^{-3} = -\frac{2t}{1+t^2}$$

MANDATORY CHECKPOINT:

$$\frac{dv}{dt} - \frac{2t}{1+t^2}V = -\frac{2t}{1+t^2} \leftarrow \text{LINEAR}$$

$$\mu = e^{\int -\frac{2t}{1+t^2}dt} = e^{\int -\frac{du}{u}} = e^{-\ln|u|} = \frac{1}{|u|} = \frac{1}{1+t^2} \quad \ldots$$

$u = 1+t^2$
$du = 2t\,dt$